

Local response of topological order to an external perturbation

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We study the behaviour of the Rényi entropies for the toric code subject to a variety of different perturbations, by means of 2D DMRG and analytical methods. We find that Rényi entropies of different index α display derivatives with opposite sign. This splitting happens at a low value of α , as opposed to typical symmetry breaking states, and can be detected on a very small subsystem regardless of the correlation length. This phenomenon is due to the presence in the phase of a point with flat entanglement spectrum, zero correlation length, and area law for the entanglement entropy. We argue that this kind of splitting is common to all the phases with a certain group theoretic structure, including quantum double models, and cluster states. The fact that the size of the subsystem does not need to scale with the correlation length, makes it possible for this effect to be accessed experimentally.

The understanding of collective behaviour arising from microscopic interactions is one of the central issues in physics. Indeed, the most recent research has been indicating that the kind of emerging order in extended systems is not so simple as was expected. This is the case of quantum phases of matter with the so-called topological order. These phases, and the phase transitions between them, cannot be characterized within the symmetry breaking mechanism, a cornerstone in many-body physics. Topological phases are characterized by global correlations that are captured by subtle entanglement properties. This makes topological order an elusive kind of order, and therefore very difficult to detect. Here we show that the order in a paradigmatic class of topological states can be revealed by measurements on a small portion of the system through the analysis of their response to an external perturbation. To this aim, we employ analytical and numerical methods in quantum many-body theory and quantum information. Our results can provide a venue to detect topological order experimentally.

The Landau symmetry breaking mechanism classifies different phases of many-body systems according to the symmetry that the system-low-energy-states breaks [1]. This implies that some observable exists whose values tell us in what phase the system is, the so called order parameter. In this picture, the macroscopic order arises from local order. For instance, we can distinguish a magnet from a paramagnet by measuring the magnetization resulting from the breaking of a local rotation symmetry of the atomic spin.

In the last two decades, a more complex scenario emerged.

An example of paramount importance in condensed matter physics is provided by quantum phase transitions occurring in two dimensional electron gas displaying fractional quantum Hall effect[2]: Different Hall states are characterized by different physical properties, despite sharing the same symmetries. Different types of topological orders have been found in various physical systems so far, like spin liquids [3], anyonic systems[4], topological insulators[5]. Since these states do not break any symmetry, they cannot be distinguished by measuring a local order parameter. The comprehension of topological phases is one of the central issues in modern condensed matter[4]. Topological states have properties like protected edge modes or degeneracies that are robust under arbitrary perturbations[4]. They possess anyonic excitations whose interactions are of topological nature and are inherently robust against decoherence. These features have made such states very interesting as a platform for quantum computers[6].

As a result from the interaction between condensed matter and quantum information, it has been understood that topological order in the ground state of a given system can be characterized through entanglement[7–9]. Specifically, a long range pattern of entanglement results in the ground state, called topological entanglement entropy γ [9]. Such a quantity can be computed through the von Neumann entropy S_1 of the reduced density matrix ρ_A describing the subsystem A (being the system $A \cup B$ in pure state). For a regular region A then von Neumann entropy is given by $S_1 = a|\partial A| - \gamma + g$, where $|\partial A|$ measures the boundary of A . The quantities a, g

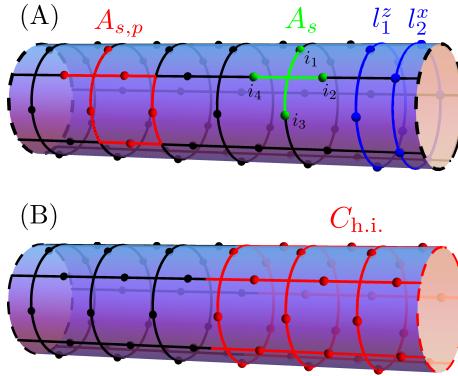


FIG. 1. Cylinder of infinite length and width $L_y = 5$ used in 2D DMRG calculation. (A) Subsystems on which Rényi entropies are calculated: A_s - one star and $A_{s,p}$ - composition of star and plaquette. Loops l_1^z and l_2^x used to distinguish between topological sectors are also depicted. (B) Subsystem $C_{h.i.}$ that contains half of the infinite cylinder.

are non universal and depend on the details of the Hamiltonian describing the system, like the correlation length ξ . The finite correction γ , instead, is universal and can serve as partial classification of topological phases[10]. We remark that, although A is a finite region, S_1 is not a local order parameter, because it is not the expectation value of a local observable. On the other hand, the reconstruction of ρ_A by local measurements seems to be quite a daunting task because A , though finite, scales with ξ [9]. Larger system sizes can be studied numerically by using Rényi entropies through the replica trick[11].

Much effort has been devoted to analyze the topological order in the ground state for a fixed value of the control parameter λ . In this paper, we push forward the idea that progress in probing topological order can be achieved by analyzing *local variations* of the ground state by external perturbation.

We study a set of spin-1/2 localized at the edges of 2D square lattice with periodic boundary conditions in presence of a perturbation V :

$$\mathcal{H} = - \sum_s \prod_{i \in s} \sigma_i^x - \sum_p \prod_{i \in p} \sigma_i^z + V(\lambda) \quad (1)$$

where s and p label the vertices and plaquettes of the lattice respectively, while σ_i^x, σ_i^z are Pauli operators of the spin living at the edge i . For $V(\lambda) = 0$ the Hamiltonian above is the celebrated toric code, a paradigmatic model for topological order[12]. For the analysis below, we remark that in this case, the ground state of this model features $\xi = 0$. We consider different $V(\lambda)$ (see Table I) where λ stands for $\{\lambda_1, \dots, \lambda_N\}$ that are the parameters controlling the perturbation. The perturbation in (1) such that the correlation length is increasing with λ_i until divergence at the critical point λ_c .

For $\lambda < \lambda_c$ these systems are topologically ordered, while for $\lambda > \lambda_c$ they are trivial paramagnets. In both phases there is no local order parameter. This model belongs to a class of so-called quantum double models which correspond to those phases whose low energy theory is a lattice gauge theory [12].

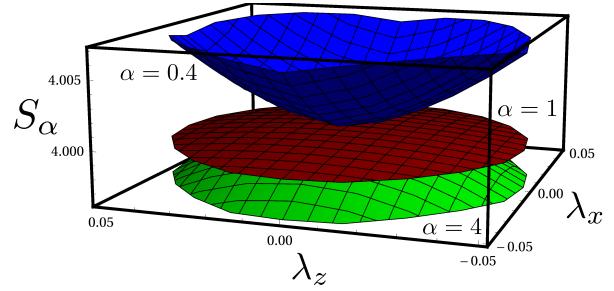


FIG. 2. The splitting phenomenon. The figure displays the splitting with opposite slopes between the small and large α Rényi entropies. We see the splitting occurring around $\alpha \simeq 0.6$. The Rényi entropies are calculated for the partition $A_{s,p}$ of Fig.1A for the ground state of $\mathcal{H} = \mathcal{H}_{TC} + V_{xz}$.

In this article, we demonstrate how the topological phase can be distinguished from the paramagnet of (1) through the inspection of $\partial_\lambda S_\alpha$, where $S_\alpha \doteq (1 - \alpha)^{-1} \log \text{Tr} \rho_A^\alpha$ are the Rényi entropies. We shall see that $\partial_\lambda S_\alpha$ present a peculiar behavior even for a small subsystem A as ξ varies as an effect of a perturbation. We remind that $S_0 = \log R$, where R is the Schmidt rank of the state, i.e. the number of nonzero eigenvalues of ρ_A , and that S_1 is the von Neumann entropy measuring the entanglement entropy for the subsystem A . We observe that, generically, R and thus S_0 increase with ξ because more degrees of freedom get entangled by increasing the room for correlations. On the other hand, we find that the behaviour of S_1 is drastically different between topological phases and the topologically trivial phases presented here.

Our approach elaborates on the quantum information notion of differential local convertibility (DLC), which states that two proximal bipartite pure states are locally convertible if it is possible to transform one into the other by resorting only on local quantum operations on A and B separately, plus classical communication and the aid of an ancillary entangled system[13]. The necessary and sufficient condition for that is $\partial_\lambda S_\alpha < 0$ for all $\alpha > 0$ [14]. In Ref.[15] such notion has been applied to the order-disorder quantum phase transition occurred in a one dimensional spin system.

For each $V(\lambda)$ we compute the ground state wavefunction $|\psi(\lambda)\rangle$ and its reduced density matrix $\rho_A(\lambda)$. For some $V(\lambda)$ we can apply exact analytical approach; for the generic perturbation $V_{xz}(\lambda) = \sum_i (\lambda_z \sigma_i^z + \lambda_x \sigma_i^x)$ we resort to numerical analysis.

The numerical method employed here is an infinite DMRG algorithm [29] in two dimensions, detailed in [26]. The method provides Matrix Product State (MPS) representation of a complete set of ground states on a cylinder of infinite length and finite width L_y (Fig.1) for a given Hamiltonian that realizes topological order. As argued in [28], each ground state has a well-defined flux threading through the cylinder. The flux is measured by (in general) dressed Wilson loop operators that enclose the cylinder in the vertical direction.

In the case of fixed point toric code (Eq.1 with $V = 0$), these loops are given by l_1^z and l_2^x (Fig.1A). Four topological

Perturbation $V(\lambda)$	G.I.	DLC	Exact	ξ
$\sum_s e^{-\lambda_s \sum_{i \in s} \sigma_i^z}$	✓	✓	✓	0
$\lambda_h \sum_{i \in H} \sigma_i^z$	✓	✗	✓	$\neq 0$
$\lambda_z \sum_i \sigma_i^z$	✓	✗	✗	$\neq 0$
$\lambda_z \sum_i \sigma_i^z + \lambda_x \sum_j \sigma_j^x$	✗	✗	✗	$\neq 0$

TABLE I. Local Convertibility in Topological Phase. DLC, i.e no splitting of the Rényi entropies, only occurs if the perturbation is fine tuned in order to keep the system with $\xi = 0$. The left column shows the type of perturbation studied. The first column details whether the considered model is Gauge Invariant. The second column indicates whether DLC occurs. For certain perturbations the ground state of the system is accessible exactly (third column). The last column provides the information on ξ . The subsystems we refer here have a non trivial bulk[21].

sectors are then distinguished by $\langle l_1^z \rangle, \langle l_2^x \rangle = \pm 1$. Once the perturbation is present, Wilson loops may change, but as long as the perturbation is small, $\langle l_1^z \rangle$ and $\langle l_2^x \rangle$ can still be used to identify topological sectors because $\langle l_1^z \rangle, \langle l_2^x \rangle \simeq \pm 1$.

Simulations are carried out with cylinders of width up to $L_y = 5$ for $\sqrt{\lambda_x^2 + \lambda_z^2} \leq 0.05$ and $0 \leq \lambda < 0.7$ as shown in Fig.2 and Fig.3 respectively. In the topological phase, the outcome of each simulation is four quasi-degenerate ground states, from which the one with $\langle l_1^z \rangle, \langle l_2^x \rangle \simeq +1$ is chosen for further investigation. This is done to ensure that finite size effects have the least possible impact on results. In the limit $L_y \rightarrow \infty$ all four ground states become locally indistinguishable. The results are converged in bond dimension of MPS which acts as a refinement parameter. A reduced density matrix of a half-infinite cylinder $C_{\text{h.i.}}$ (Fig.1B) is calculated throughout the simulation. The bond dimension is increased until convergence of its spectrum is reached.

In Fig.2, we can see the behaviour of the S_α Rényi entropies as we span the parameter space λ_x, λ_z for the perturbation V_{xz} . We see clearly that in the topologically ordered phase a splitting of S_α 's occur: $\partial_\lambda S_\alpha \lesssim 0$ at a given value of $\alpha = \alpha_0$; we found $\alpha_0 \simeq 0.6$ (see caption of 2). We call this phenomenon low- α splitting in the rest of this article. In the paramagnetic phase all the Rényi entropies are monotones with λ . This behaviour is generically independent of the size and shape of the subsystem A , as long as A contains some bulk [21]. Below we provide an explanation of the phenomenon. The topologically ordered phase we consider is characterized by the presence of a state (at $\lambda = 0$) with $\xi = 0$ and a flat entanglement spectrum (and an area law)[10]. The flat entanglement spectrum implies that small perturbations results in decreasing S_α for $\alpha > \alpha_0$ being $\alpha_0 < 1$, because the distribution becomes less flat in the most represented eigenvalues in the entanglement spectrum. In contrast, S_0 must increase with ξ as an effect of the perturbation (new degrees of freedom are involved in the entanglement spectrum). So the low- α splitting results from the insertion of a finite ξ in the state evolving from a state with flat spectrum and zero ξ . We also observe that such property is shared with the so called G -states which

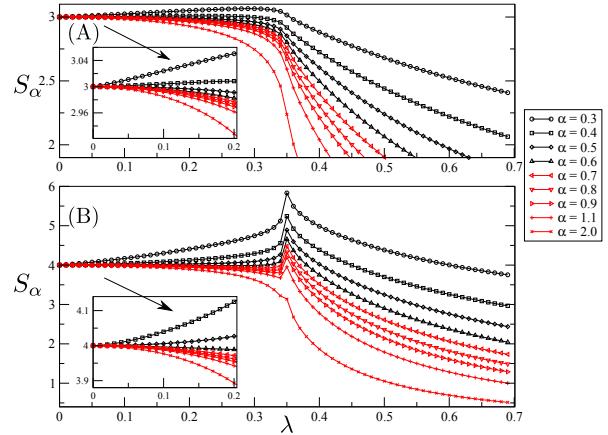


FIG. 3. Rényi entropies as a function of λ for the ground state of $\mathcal{H} = \mathcal{H}_{TC} + V_{xz}$ with $\lambda_x = \lambda$ and $\lambda_z = \lambda/2$. Here, $L_y = 5$. The reduced system A consists of A_s and $C_{\text{h.i.}}$ in panels A and B, respectively. As λ increases, the correlation length increases. The Schmidt rank R and the low $\alpha < \alpha_0$ -Rényi entropies increase as well. The value of α_0 are 0.4, 0.6 in panel A and B, respectively. Nevertheless, the total entanglement S_1 and all the higher Rényi entropies are decreasing with ξ . Notice the spike in panel B marking the quantum phase transition to the paramagnetic phase at $\lambda_c \sim 0.35$.

include all the topologically ordered quantum double models, and states like the cluster states[27], and therefore our findings apply to this class of models as well[10] (see [31] for a discussion of the cluster phase diagram). For the corresponding ground states the splitting is thus also expected to occur at $\alpha_0 < 1$ no matter how we perturb in the plane λ_x, λ_z . Note again that in the paramagnetic phase all the $\partial_\lambda S_\alpha$ have the same sign and no splitting ever occurs, which is easily understood from the presence (at very large λ) of a completely factorized state, see Fig.3.

Now we argue how the specific values at which the splitting phenomenon occurs effectively distinguish the topologically ordered state from a topologically-trivial ordered state (like a ferromagnet). Indeed, as discussed above, the latter states have typically S_1 increasing with ξ (S_2 is also increasing as well as long as ξ is not too large). Accordingly, the splitting phenomenon is shown to occur at higher values of α for phases with broken symmetries[15]. We also observe that it may or may not occur for these phases, depending on local details of the system (the anisotropy in the case of the study in Ref.[15]).

Summarizing: We can distinguish between the topological phase and the paramagnet of (1). Furthermore, we can distinguish between the topological phase and a symmetry breaking phase because the splitting phenomenologically occurs at $\alpha > 1$ in the latter.

To corroborate our findings, we resort to exact analysis for suitable perturbations $V(\lambda)$'s. We consider two cases *i*) $V_h = \lambda_z \sum_{i \in h} \sigma_i^z$, corresponding to placing the external field $\propto \sigma^z$ only along the horizontal links of the lattice; *ii*) $V(\lambda) = \sum_s e^{-\lambda \sum_{i \in s} \sigma_i^z}$ leading to the Castelnovo-Chamon model[20]. Since these perturbations commute with the pla-

quette operators of Eq.1, the ground state of these models can be written as the superposition of loop states $|g\rangle$ with amplitudes $\alpha(g)$. A loop state $|g\rangle$ is obtained from the completely polarized state in the z direction, by flipping down all the spins intersected by a loop drawn on the lattice. The corresponding loop operators g form a group G called the gauge group of these theories.

In the case *i*) the star operators $\prod_{i \in s} \sigma_i^x$ interact only along the rows of the lattice. The model maps onto the product of arrays of Ising chains by the duality $A_s \rightarrow \tau_\mu^z$, $\sigma_i^z \rightarrow \tau_\mu^x \tau_{\mu+1}^x$: $\mathcal{H}_{TC} + V_h \mapsto \mathcal{H}_{ff} = \bigoplus_{i=1}^L (-\lambda \sum_\mu \tau_\mu^x \tau_{\mu+1}^x - \sum_\mu \tau_\mu^z)$ [17, 18]. The relevant correlators in the variables σ can be obtained through the correlators in the dual variables τ that can be accessed exactly[19]. In the following, we sketch a proof that the splitting phenomenon does occur in this model (see [21] for details). We consider the star $A_s = \{i_1, i_2, i_3, i_4\}$ as subsystem A (see Fig.1); ρ_{A_s} is block diagonal with 2×2 blocks labeled by $|i_1 i_2 i_3 i_4\rangle$ and $A_s |i_1 i_2 i_3 i_4\rangle$. It results that ρ_A has maximum rank unless $\alpha(g) = \alpha(g_1)\alpha(g_2)$, implying there is a zero eigenvalue in each block. In the dual picture this is equivalent to require $\langle \tau_i \tau_j \rangle = \langle \tau_i \rangle \langle \tau_j \rangle$. Such condition holds at $\lambda = 0$ only, and therefore R increases at $\lambda \neq 0$. The factorization of the amplitudes also proves that both the $\alpha = 1, 2$ -Rényi entropies decrease at small λ [21].

The case *ii*) is important to test the argument of the interplay between splitting and increasing of correlation length. This argument implies that a perturbation for which $\xi(\lambda) = \text{const}$ does not lead to a splitting in the Renyi entropies. The model of Castelnovo-Chamon features exactly this, since spin-spin correlation functions $\langle \sigma_i^x \sigma_j^x \rangle$ are vanishing for every value of λ . The exact ground state is made of loops with amplitudes $\alpha(g) = e^{-\lambda/2 \sum_{i \in s} \sigma_i^z(g)}$, where $\sigma_i^z(g) = \langle g | \sigma_i^z | g \rangle$. The topological phase is ($\lambda < \lambda_c \approx 0.44$). A lengthy calculation leads to $S_\alpha(\rho_A) = (1 - \alpha)^{-1} \log Z^{-\alpha}(\lambda) \sum_{g \in G} e^{-\lambda L_g} w^{\alpha-1}(\lambda, g)$, where $Z = \sum_g e^{-\lambda L_g}$ and $w(\lambda, g) := \sum_{h \in G_A, k \in G_B} e^{-\lambda L_{hgk}}$, L_m is the length of the loop m of the gauge group G ; here G_A and G_B are the gauge groups of the subsystems A and B respectively. The analysis of small and large λ expansions reveals that $\partial_\lambda S_\alpha(\lambda) \leq 0$ [21]. As a particular case, S_0 is constant for every value of λ . Accordingly, for this fine tuned perturbation all Rényi entropies decrease and therefore no splitting is observed. This is consistent with the fact that also in this model the amplitudes $\alpha(g)$ factorize as discussed in *i*).

In this paper, we have shown that topologically ordered states in the paradigmatic class of quantum double models have a peculiar way of responding to perturbations: The Rényi entropies of a subsystem display opposite slopes around $\alpha \approx 0.6$. The subsystem size can be small independently of ξ . In the paramagnetic phases, in contrast, no such a phenomenon is observed. The low- α splitting is shown to be a robust characteristic of the topological phase, see Fig.2. In a quantum information context, we can say that, while paramagnets are always differential locally convertible, the class of topologically ordered ground states we studied here are not.

We close this paper with some speculations and future di-

rections. Comparing our findings with the study in [15], we see that the kind of Rényi entropies splitting that may occur in symmetry-breaking-phases and the one characterizing topological states are phenomenologically very different. We conjecture that in topologically-trivial ordered phases the low- α -splitting can only be accidental, meaning that every symmetry breaking state can be continuously "deformed" without observing it. Therefore, by measuring ξ together with the von-Neumann entropy S_1 for a small subsystem (it can be shown this is sufficient to show whether low- α splitting is present), one could tell apart topological states of the class studied here from symmetry breaking states. This should be contrasted with the standard approach based on topological entropy where the size of the subsystem must be much larger than ξ .

Our results may provide the basis for the experimental detection of exotic phases like topologically ordered or cluster states. In this endeavour, it would be desirable to investigate these findings in presence of disorder.

It would be also important to investigate the behaviour of Rényi entropies under perturbations for more general topological states without a flat entanglement spectrum like fractional quantum Hall liquids and chiral spin liquids[11, 24], and for symmetry protected topological states like one dimensional systems or topological insulators[25]. A relevant question is whether the splitting behaviour can shed light on the finite temperature resilience of topological order, and on the very existence of anyonic excitations.

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